Simple plane-filling curves: the root-2 and root-3 families

Herman Haverkort

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1 Introduction

One of the many ways in which plane-filling curves can be described is the following, used by Ventrella[1]. One starts with a line segment; the segment is directed (it has a head and a tail) and oriented (its left and right side are distinct: one side is marked). We indicate the direction and orientation by an arrow head on the marked side of the head. A replacement rule describes how such a line segment is replaced by a polyline (a chain) of directed, oriented line segments. Arrow heads on the segments of the polyline help to determine how each segment is obtained from the original line segment by scaling, translation, rotation, and/or reflection (see Figure 1 for an example). Applying the replacement rule recursively to each line segment results in a fractal curve. If the sum of the squared lengths of the segments of the replacing polyline is equal to the squared length of the original line segment, then the fractal curve has dimension two and, if it does not overlap itself too much, it will be a plane-filling curve.

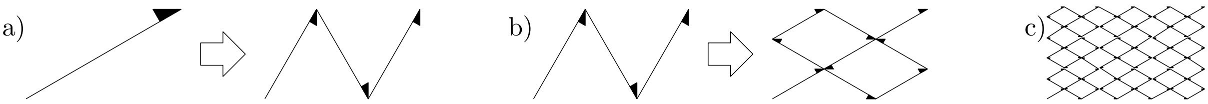


Figure 1: a) A definition of the Peano curve, stretched by a factor $\sqrt{3}$ in the horizontal dimension. b) The result of expanding the definition once. c) The result of expanding the definition two more times—the contours of the rectangular area filled by the curve start to become visible.

2 The root-2 family

By some measure, the simplest plane-filling curves are those that are defined by a replacement rule with two segments of the same length. Ventrella calls this the root-2 family. Suppose the replacement segments each have length one. Because the starting point and the end point of the chain of segments must be a distance $\sqrt{2}$ apart, the two segments must make a 90 degrees' angle with each other. We consider curves to be specimens of the same curve if a similarity transformation maps one curve to the other. Thus we may assume, without loss of generality, that the two segments make a right turn; all curves based on left turns are just reflected copies of right-turn-based curves, and we do not need to discuss them separately.

The only thing that remains to specify is where the arrow heads are. For each line segment, there are four options:

arrow position	in Ventrella's notation	short notation
at the head of the line segment, on the left	$1, 1$	$\rightharpoonup$ or q
at the head of the line segment, on the right	$1, -1$	$\rightharpoondown$ or p
at the tail of the line segment, on the left	$-1, 1$	$\leftharpoonup$ or d
at the tail of the line segment, on the right	$-1, -1$	$\leftharpoondown$ or b

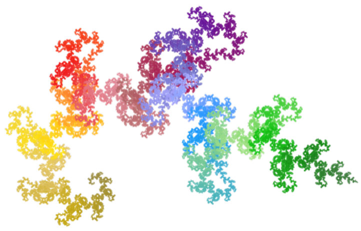
1. J. Ventrella: *Brain-filling curves—A fractal bestiary.* Eyebrain Books, 2012.

Each curve of the root-2 family is described by the pair of letters that specifies the chosen option for the first segment and the chosen option for the second segment. Some of these letter pairs define symmetric curves. Each symmetric curve is defined by four equivalent letter pairs, since for symmetric curves, reversing a segment (moving the arrow head to the other end) does not have any effect. Asymmetric curves are each described by two pairs of letters: one pair describes the reflected reverse of the other. Thus, in effect, their are only six different curves in the root-2 family:

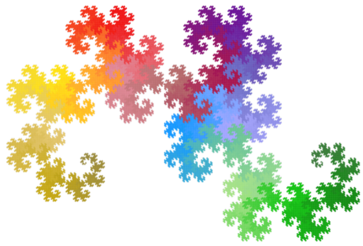
def.	name
qq	**Lévy C-curve** (symmetric)
qp	**"birds"**
qd	→ *identical to* qq
qb	**Harter-Heighway Dragon**
pq	→ *reflected reverse of* qp
pp	**Pólya curve (Sierpiński curve)** (symmetric)
pd	**alternative isosceles-right-triangle sweep** (not to be confused with pp, which fills the same shape in a different way)
pb	→ *identical to* pp
dq	→ *identical to* qq
dp	→ *reflected reverse of* pd
dd	→ *identical to* qq
db	**"sponge"**
bq	→ *reflected reverse of* qb
bp	→ *identical to* pp
bd	→ *reflected reverse of* db
bb	→ *identical to* pp

Below, these curves are depicted with a colour gradient, such that each curve changes colour from brownish, via yellow, red, purple, and blue to green as it twists its way from the beginning to the end:

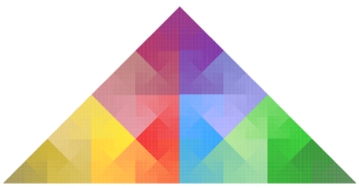
qq Levy, Cesaro
related: qd,dq,dd



qp "birds"
related: pq

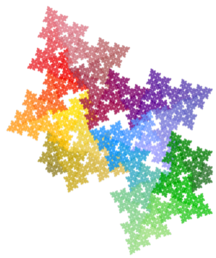


qb Harter-Heighway Dragon
related: bq



pp Pólya, Sierpiński, Knopp
related: pb,bp,bb

pd
related: dp



db "sponge"
related: bd

It is not immediately obvious that all of these curves are plane-filling curves. For qb, pp and pd this is well-known and, especially for the triangle sweeps, it is relatively easy to see, as the curve follows a recursive tessellation that is relatively easy to recognize. But what about the other curves? By definition, a curve is plane-filling if its image (the set of points visited by the curve) has positive two-dimensional Jordan content. This is equivalent to saying that somewhere, there must be a square that is entirely covered by the curve.

For the Lévy curve this can be shown as follows. We associate each oriented segment h in the recursive construction with an isosceles right triangle T, whose interior lies to the left of h, and of which h is the hypothenuse; conversely, we consider the hypothenuse h of any right triangle T to be directed, such that the interior of T lies to the left of h. Let G_0 be a plane-filling grid of unit squares, each subdivided into four right isosceles triangles that meet in the centre of the square, such that the initial line segment in the curve definition is one of the long edges (hypothenuses) in this grid. For $i = 1, 2, 3, ...$, let G_i be the grid of triangles obtained by cutting each triangle of G_{i-1} along the bisector of its right angle.

The definition of the Lévy curve establishes a one-to-two correspondence between triangles of G_{i-1} and G_i: when T is a triangle in G_{i-1}, then T' and T'' are the triangles of G_i whose directed hypothenuses are obtained from applying the segment rewriting rule to the directed hypothenuse of T; conversely, for each triangle T in G_i, there is exactly one triangle T° in G_{i-1} such that the hypothenuse of T can be obtained by applying the segment rewriting rule to the hypothenuse of T°. Furthermore, observe that if we start from a segment h and apply the segment rewriting rule to the limit, we will not reach any points at distance more than $|h|$ (the length of h) from h.

If we now sketch the Lévy curve by applying the rewriting rule 16 times (using a computer program), starting from a line segment of length 1, we find at least one square Q that has the following properties: Q has side length $(\sqrt{2})^{-16}$, and all 24 directed hypothenuses in G_{16} that have at least one end point on Q, are part of the curve. Now, by the previous observations, for each point q inside Q there is a sequence Σ of triangles $T_{16}, T_{17}, T_{18}, ...$ with directed hypothenuses $h_{16}, h_{17}, h_{18}, ...$ in $G_{16}, G_{17}, G_{18}, ...$, respectively, with the following properties:

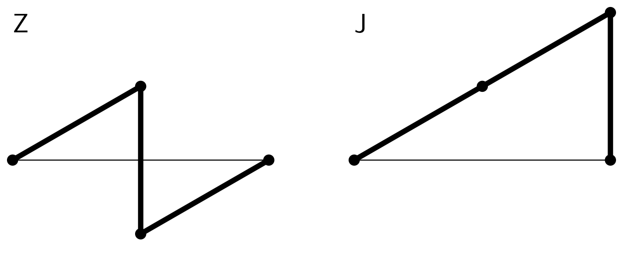
- h_{16} is one of the 24 directed hypothenuses that have a vertex on Q;
- for $i > 16$, each h_i is one of the segments obtained by applying the segment rewriting rule to h_{i-1};
- Σ converges to q.

Thus, if we keep applying the segment rewriting rule, then, in the limit, each point q in Q is covered by the curve.

The “birds” and “sponge” curves can be shown to be plane-filling in a similar way, except that we consider a sequence of grids in which each triangle is covered twice; once with the hypothenuse directed such that the triangle lies to the left; and once with the hypothenuse directed such that the triangle lies to the right[2]. Thus, around a square Q, we consider 48 directed and oriented hypothenuses: each hypothenuse is visited four times (in each of two directions, for each of two adjacent triangles). Otherwise the proof is the same.

3 The root-3 family, triangle-grid subfamily

The root-3 family consists of curves whose segment rewriting rule consists of a chain of three segments of length 1, such that the starting point and the end point are a distance $\sqrt{3}$ apart. Modulo symmetries, we can distinguish two possible genera in which the segments lie on a grid of equilateral triangles: the zigzag (Z) genus, where the chain makes a 120 degrees’ turn to the right, followed by a 120 degrees’ turn to the left; and the hook (J) genus, where the chain first goes straight ahead over two segments, and then turns 120 degrees to the right:



For each genus, there are, in principle, $4^3 = 64$ ways to put arrowheads on the segments. However, after expanding the definition k times for some number k, many of these curves contain the same segment

2. Alternatively, we can cut each triangle in two subtriangles; one to be covered by the hypothenuse in one direction; the other to be covered in the other direction. Thus, each triangle in the grid is covered only once, but associating triangles with segments in this grid is slightly more complicated.

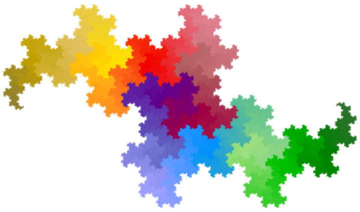
with the same arrowhead twice. This reduces the number of *different* segments on level k to less than 3^k. If such a curve would be plane-filling, then the area filled by the curve would have to be some positive constant c times the squared scale factor. By the above, we would have $c \cdot 1^2 < 3^k \cdot c \cdot ((\sqrt{3})^{-k})^2 = c$, which is not possible. Therefore, if, at some level of expansion, the same segment with the same arrowhead occurs twice, then the curve cannot be plane-filling.

For each Z- or J-curve for which I did *not* find a duplicate segment after applying the segment rewriting rule eight times, I could establish a matching one-to-three correspondence between grids of triangles or related shapes, and I also found a (sometimes very small) triangle completely covered by the curve—thus proving that the curve is plane-filling.

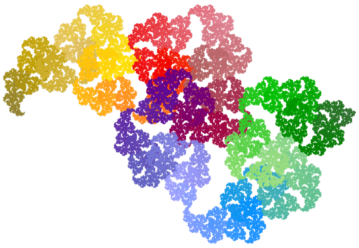
Among the Z-curves, there are essentially 20 different curves, 12 of which are plane-filling:

root-3 family, triangular-grid subfamily, zigzag (Z) genus

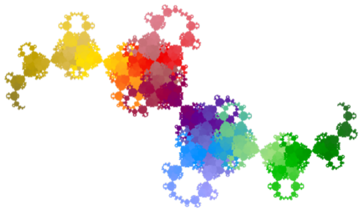
def.	name	plane-filling?	def.	name	plane-filling?
Zqqq	**Knuth's Terdragon**	**yes**	Zdqq	→ *reflected reverse of* Zqqd	
Zqqp	**"cloud"**	**yes**	Zdqp	→ *identical to* Zpqp	
Zqqd		no	Zdqd	→ *identical to* Zpqp	
Zqqb	→ *identical to* Zqqq		Zdqb	**"palace"**	**yes**
Zqpq	**"Peano ballet"**	**yes**	Zdpq	→ *reflected reverse of* Zqpd	
Zqpp	*(no name)*	**yes**	Zdpp	→ *identical to* Zppp	
Zqpd		no	Zdpd	→ *identical to* Zppp	
Zqpb	→ *identical to* Zqpq		Zdpb	*(no name)*	**yes**
Zqdq	→ *identical to* Zqpq		Zddq	→ *reflected reverse of* Zqdd	
Zqdp	**"fountain"**	**yes**	Zddp	→ *identical to* Zppp	
Zqdd		no	Zddd	→ *identical to* Zppp	
Zqdb	→ *identical to* Zqpq		Zddb	*(no name)*	**yes**
Zqbq	→ *identical to* Zqqq		Zdbq	→ *reflected reverse of* Zqbd	
Zqbp	**"claw"**	**yes**	Zdbp	→ *identical to* Zpqp	
Zqbd		no	Zdbd	→ *identical to* Zpqp	
Zqbb	→ *identical to* Zqqq		Zdbb	*(no name)*	**yes**
Zpqq	→ *reflected reverse of* Zqqp		Zbqq	→ *identical to* Zqqq	
Zpqp	**Peano (stretched)**	**yes**	Zbqp	→ *reflected reverse of* Zpqb	
Zpqd	→ *identical to* Zpqp		Zbqd	→ *reflected reverse of* Zdqb	
Zpqb		no	Zbqb	→ *identical to* Zqqq	
Zppq	→ *reflected reverse of* Zqpp		Zbpq	→ *identical to* Zqpq	
Zppp	**"butterfly"**	**yes**	Zbpp	→ *reflected reverse of* Zppb	
Zppd	→ *identical to* Zppp		Zbpd	→ *reflected reverse of* Zdpb	
Zppb		no	Zbpb	→ *identical to* Zqpq	
Zpdq	→ *reflected reverse of* Zqdp		Zbdq	→ *identical to* Zqpq	
Zpdp	→ *identical to* Zppp		Zbdp	→ *reflected reverse of* Zpdb	
Zpdd	→ *identical to* Zppp		Zbdd	→ *reflected reverse of* Zddb	
Zpdb		no	Zbdb	→ *identical to* Zqpq	
Zpbq	→ *reflected reverse of* Zqbp		Zbbq	→ *identical to* Zqqq	
Zpbp	→ *identical to* Zpqp		Zbbp	→ *reflected reverse of* Zpbb	
Zpbd	→ *identical to* Zpqp		Zbbd	→ *reflected reverse of* Zdbb	
Zpbb		no	Zbbb	→ *identical to* Zqqq	



Zqqq Terdragon
related: Zqqb,Zqbq,Zqbb,Zbqq,Zbqb,Zbbq,Zbbb,Jqqb,Jqbq,Jbqq

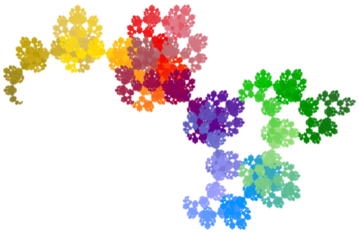


Zqqp "cloud"
related: Zpqq

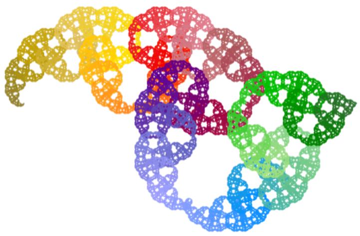


Zqpq "Peano ballet"
related: Zqpb,Zqdq,Zqdb,Zbpq,Zbpb,Zbdq,Zbdb,Jqbp

Zqpp (no name)
related: Zppq

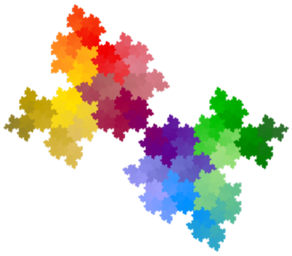


Zqdp "fountain"
related: Zpdq,Jdqp

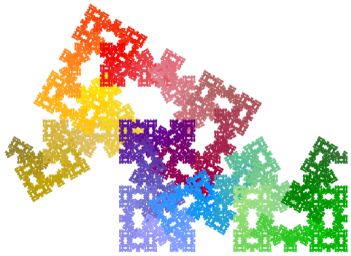


Zqbp "claw"
related: Zpbq,Jqdb

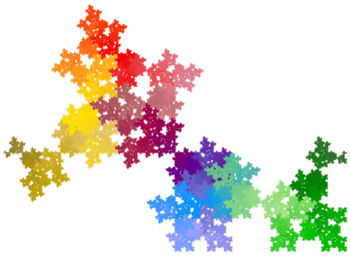
Zpqp Peano (stretched)
related: Zpqd,Zpbp,Zpbd,Zdqp,Zdqd,Zdbp,Zdbd,Jpdq



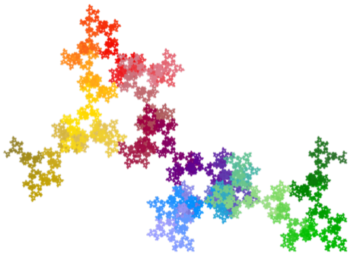
Zppp "butterfly"
related: Zppd,Zpdp,Zpdd,Zdpp,Zdpd,Zddp,Zddd,Jpdp,Jpdd,Jddp



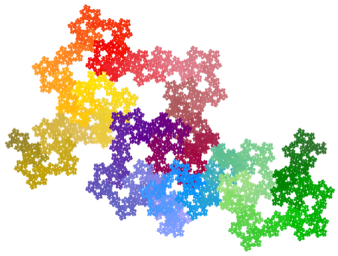
Zdqb "palace"
related: Zbqd



Zdpb (no name)
related: Zbpd



Zddb (no name)
related: Zbdd



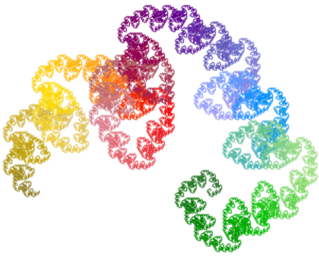
Zdbb (no name)
related: Zbbd

Among the J-curves, there are 12 curves that are equivalent to (part of) another curve that succeeds it in the lexicographical order[3]; among the remaining curves, 28 are plane-filling, 24 are not.

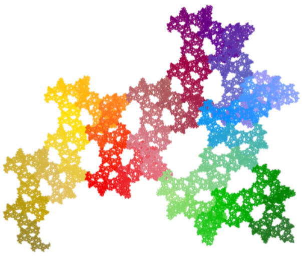
root-3 family, triangular-grid subfamily, hook (J) genus

def.	name	plane-filling?	def.	name	plane-filling?
Jqqq	"lace"	**yes**	Jdqq		no
Jqqp	*(no name)*	**yes**	Jdqp	→ *identical to reflected last 2/3 of* Zqdp	
Jqqd		no	Jdqd	*(no name)*	**yes**
Jqqb	→ *identical to first 2/3 of* Zqqq		Jdqb	*(no name)*	**yes**
Jqpq	*(no name)*	**yes**	Jdpq		no
Jqpp	"Zealand dragon"	**yes**	Jdpp		no
Jqpd		no	Jdpd	"Peano stripes"	**yes**
Jqpb		no	Jdpb	"Peano railroads"	**yes**
Jqdq	"curl"	**yes**	Jddq		no
Jqdp	"magic mushroom"	**yes**	Jddp	→ *identical to first 2/3 of* Zppp	
Jqdd		no	Jddd	"shield"	**yes**
Jqdb	→ *identical to first 2/3 of* Zqbp		Jddb	*(no name)*	**yes**
Jqbq	→ *identical to half of* Zqqq		Jdbq		no
Jqbp	→ *identical to half of* Zqpq		Jdbp		no
Jqbd		no	Jdbd	"forest"	**yes**
Jqbb		no	Jdbb	"leaves"	**yes**
Jpqq	*(no name)*	**yes**	Jbqq	→ *identical to last 2/3 of* Jqbq	
Jpqp	"sail"	**yes**	Jbqp		no
Jpqd	→ *reflected reverse of last 2/3 of* Jqdp		Jbqd	"foam"	**yes**
Jpqb		no	Jbqb	*(no name)*	**yes**
Jppq	*(no name)*	**yes**	Jbpq		no
Jppp	"ice cream"	**yes**	Jbpp		no
Jppd		no	Jbpd	"beetlefrog"	**yes**
Jppb		no	Jbpb	"crab"	**yes**
Jpdq	→ *identical to half of* Zpqp		Jbdq	→ *identical to last 2/3 of* Jpbq	
Jpdp	→ *identical to half of* Zppp		Jbdp		no
Jpdd	→ *reflected reverse of last 2/3 of* Jpdp		Jbdd	*(no name)*	**yes**
Jpdb		no	Jbdb	*(no name)*	**yes**
Jpbq	"Peano waters"	**yes**	Jbbq		no
Jpbp	"tripolya"	**yes**	Jbbp		no
Jpbd		no	Jbbd	"crystal"	**yes**
Jpbb		no	Jbbb	*(no name)*	**yes**

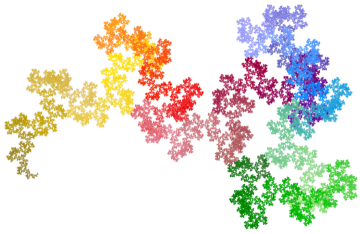
3. This includes all 7 J-curves in Ventrella's book, which are ultimately part of Zqqq, Zpqp, or Zppp.



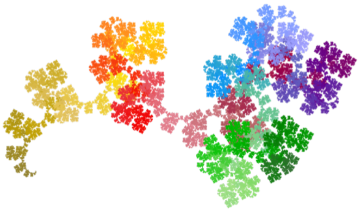
Jqqq “lace”
related: –



Jqqp
related: –



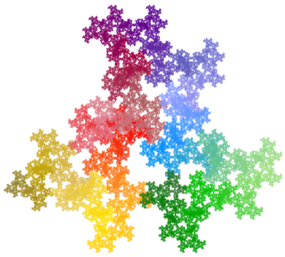
Jqpq
related: –



Jqpp “Zealand dragon”
related: –

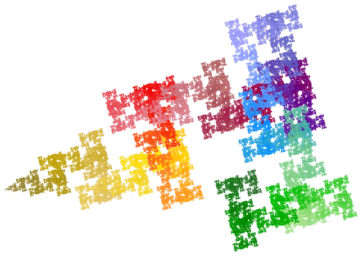
Jqdq “curl”
related: –

Jqdp “magic mushroom”
related: Jpqd

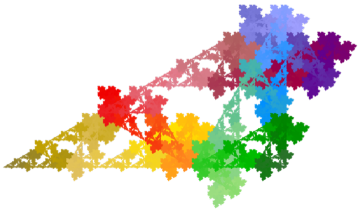


Jpqq
related: –

Jpqp “sail”
related: –

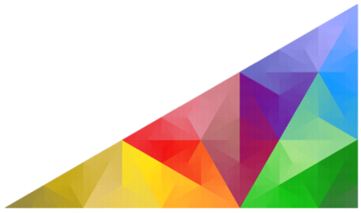


Jppq
related: –



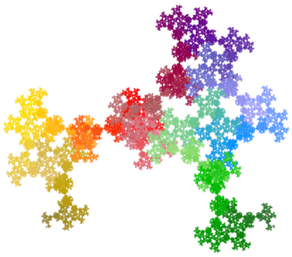
Jppp “ice cream”
related: –

Jpbq “Peano waters”
related: Jbdq



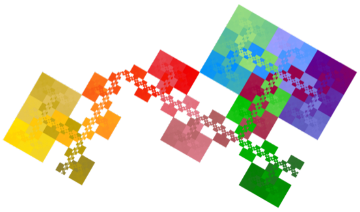
Jpbp “tripolya”
related: –

Jdqd
related: –



Jdqb
related: –

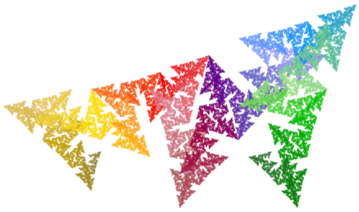
Jdpd “Peano stripes”
related: –



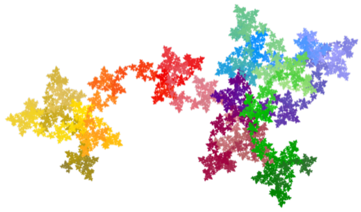
Jdpb “Peano railroads”
related: –

Jddd “shield”
related: –

Jddb
related: –

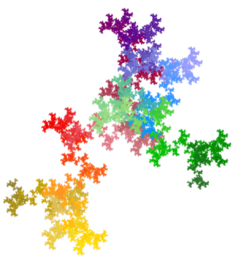


Jdbd “forest”
related: –

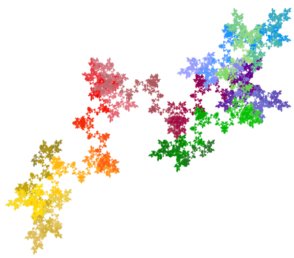


Jdbb “leaves”
related: –

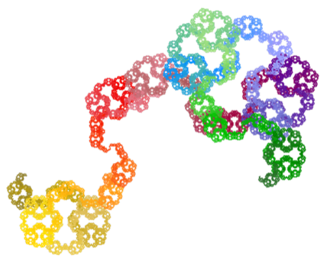
Jbqd “foam”
related: –



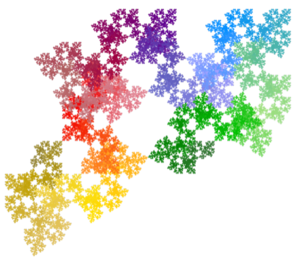
Jbqb
related: –



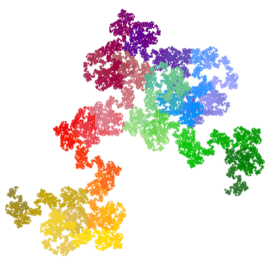
Jbpd “beetlefrog”
related: –



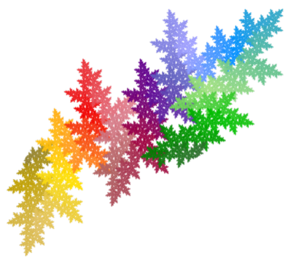
Jbpb “crab”
related: –



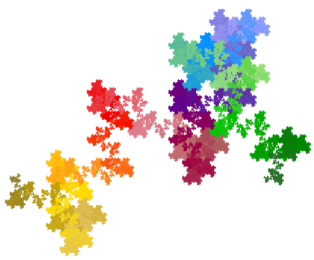
Jbdd
related: –



Jbdb
related: –



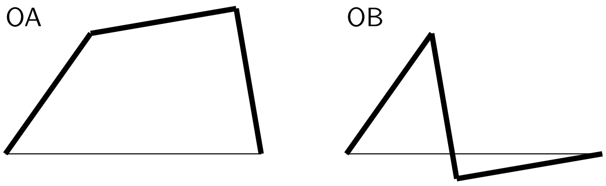
Jbbd “crystal”
related: –



Jbbb
related: –

4 The root-3 family, octilinear subfamily

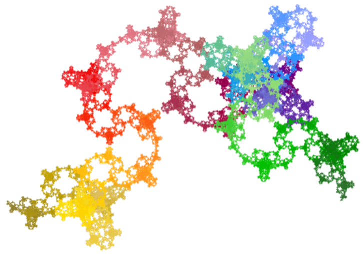
There are other ways to replace a line segment of length 1 by a chain of three segments of length $1/\sqrt{3}$ each. For example, we could use line segments that make angles that are multiples of 45 degrees—there are three essentially different ways to do that:



Each of these three patterns allows 64 options for placing the arrowheads. Some of these result in beautiful curves, but I do not know whether they are plane-filling curves. The proof technique that was used above for the Lévy curve seems hard to apply, because now, the segments do not stick to the edges of a regular grid. Some examples of curves based on patterns with 45, 90 and 135 degrees' angles are the following:

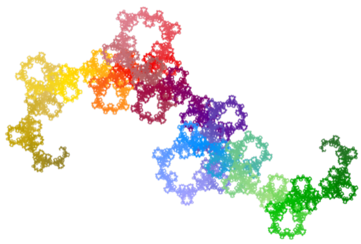
OAqdq
related: ?

OApqd
related: ?



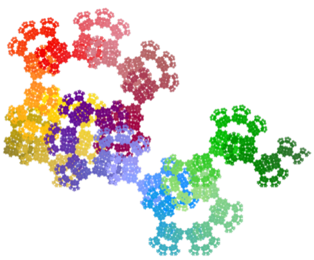
OAbpb
related: ?

OBqdq
related: ?



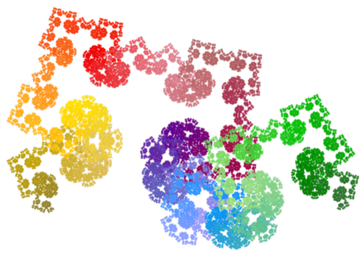
OBqdb
related: ?

OBpbq
related: ?



OBpbp
related: ?

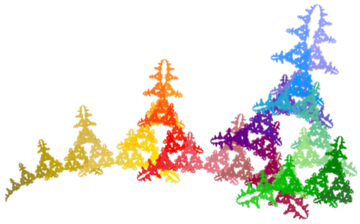
OBpbd
related: ?



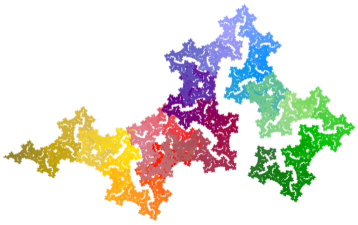
OBdqd
related: ?

OBdbp
related: ?

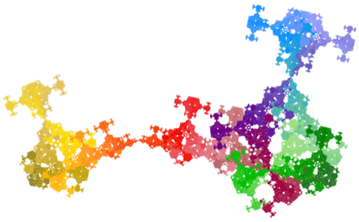
OCqpb
related: ?



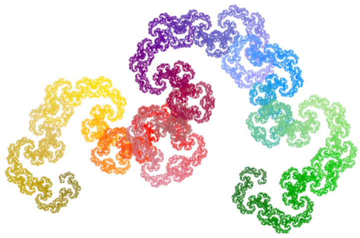
OCqbp
related: ?



OCpbq
related: ?



OCdbp
related: ?



OCdqq
related: ?

OCbpb
related: ?