The Rauzy triangle plane-filling curve

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Because it seems hard or impossible to find elsewhere, here is a description of a plane-filling curve that fills half of the Rauzy fractal [1]. I found the underlying tessellation on Stewart Hinsley's website in 2013, which has since been archived [2]. Given the tessellation, the plane-filling curve is obvious. I call it the *Rauzy triangle curve*, although *Rauzy quadrilateral curve* might be more accurate. Here it is.

The tessellation can be constructed from three types of tiles. Let χ be the Tribonacci constant, that is, $\chi \approx 1.83929$ is the real root of $\chi^3 - \chi^2 - \chi - 1$. Let c be $\sqrt{\chi}$. Let α be $\arctan(\sqrt{1 + 2/\chi}) \approx 55.311°$. On refinement level i of the recursive tessellation we will find, as illustrated in Figure 1:

- quadrilaterals with side lengths and angles (see the figure for $i = 0$):
  $c^{-i}, \pi - 2\alpha, c^{-i-2}, \pi - \alpha, c^{-i-3}, 2\alpha, c^{-i-1}, \alpha$;
- hexagons with side lengths and angles (see the figure for $i = 1$):
  $c^{-i-2}, \pi - \alpha, c^{-i-3}, \pi + \alpha, c^{-i-2}, \alpha, c^{-i-1}, \pi - \alpha, c^{-i-2}, 2\alpha, c^{-i}, \pi - 2\alpha$;
- octagons with side lengths and angles (see the figure for $i = 2$):
  $c^{-i-1}, \pi - \alpha, c^{-i-2}, \pi + \alpha, c^{-i-1}, \alpha, c^{-i}, \pi - \alpha, c^{-i-1}, \pi - \alpha, c^{-i-2}, \pi - \alpha, c^{-i-3}, \pi + \alpha, c^{-i-2}, \alpha$.

Tiles only meet vertex-to-vertex and edge-to-edge, that is, no vertex of a tile lies in the interior of a neighbouring tile's edge. The figure also shows in which corners of a tile the plane-filling curve enters and leaves the tile.

The recursive construction starts with one quadrilateral tile at level $i = 0$. To obtain refinement level $j = i + 1$ from refinement level i, all edges of length c^{-i} are replaced by a chain of three edges with lengths $c^{-j-2}, c^{-j-3}, c^{-j-2}$, as shown in the figure. Since this chain is symmetric in the midpoint of the edge that is replaced, this refinement leaves the sizes (two-dimensional measures) of the tiles unaffected. The refinement turns quadrilaterals into hexagons, hexagons into octagons, and octagons into decagons. The decagons are subsequently subdivided into a quadrilateral, a hexagon and an octagon, as shown in the figure for $i = 3$.

A more refined sketch (without the tile boundaries) is shown in Figure 2, along with a three-dimensional path whose projection on the horizontal plane follows the plane-filling curve while steadily moving upwards in the third dimension.

A definition in Ventrella's notation [3] (modulo scaling and rotation) is given below (version A); in this notation, $i : x, y, r, r$ means that the vector from the starting point to the end point of the i-th segment of the plane-filling curve is (x, y), while $r = -1$ or $r = 1$ indicates whether the segment represents a reversed or a forward copy, respectively, of the scaled, rotated and translated curve as a whole.

Version A:	*Version B:*	*Version C:*
Square grid	Square grid	Square grid
3 segments	3 segments	3 segments
segment values:	segment values:	segment values:
1: $1, \tan(\alpha), 1, 1$	1: $-\chi, \chi\tan(\alpha), -1, -1$	1: $-\chi, \chi\tan(\alpha), -1, -1$
2: $-\chi, \chi\tan(\alpha), 1, 1$	2: $1, \tan(\alpha), -1, -1$	2: $2/(\chi - 1), 0, 1, 1$
3: $2/(\chi - 1), 0, -1, -1$	3: $2/(\chi - 1), 0, -1, -1$	3: $1, \tan(\alpha), 1, 1$

Two alternative tilings of the decagon (versions B and C) are illustrated in Figure 1, see also the corresponding definitions in Ventrella's notation that are given above. Interestingly, all three versions yield the same plane-filling curve in the end!

References

[1] https://en.wikipedia.org/wiki/Rauzy_fractal

[2] Stewart Hinsley: *Tiles Associated with the 4th Unit Cubic Pisot Number.* Personal web page, 2002, last retrieved 2 Nov 2022 from archive.org at https://web.archive.org/web/20120129083647/http://www.meden.demon.co.uk/Fractals/cubic4.html

[3] Jeffrey Ventrella: *Brain-filling curves—A fractal bestiary.* Eyebrain Books, 2012.

[4] Herman Haverkort: Plane-filling trails, in Proc. *Symp. on Comp. Geom. Media Exposition 2020*, LIPIcs 164:81:1–5; full version at *CoRR*, https://arxiv.org/abs/2003.12745.

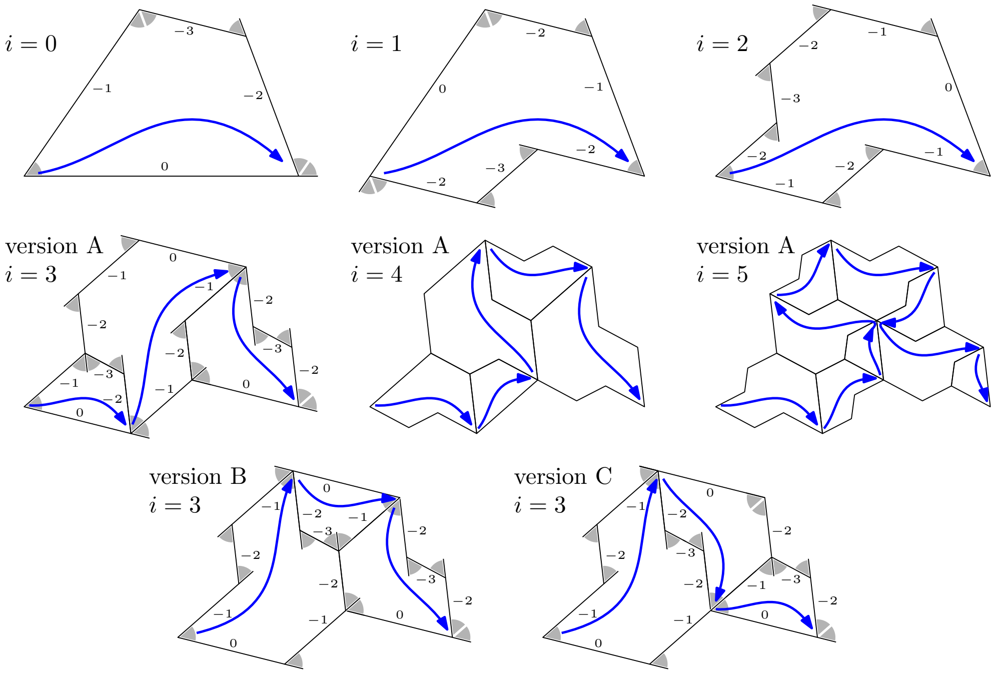

Figure 1: Graphical definition of the Rauzy triangle plane-filling curve. Edges marked with a number $-k$ have length c^{-i-k}. Each pie slice represents an angle α.

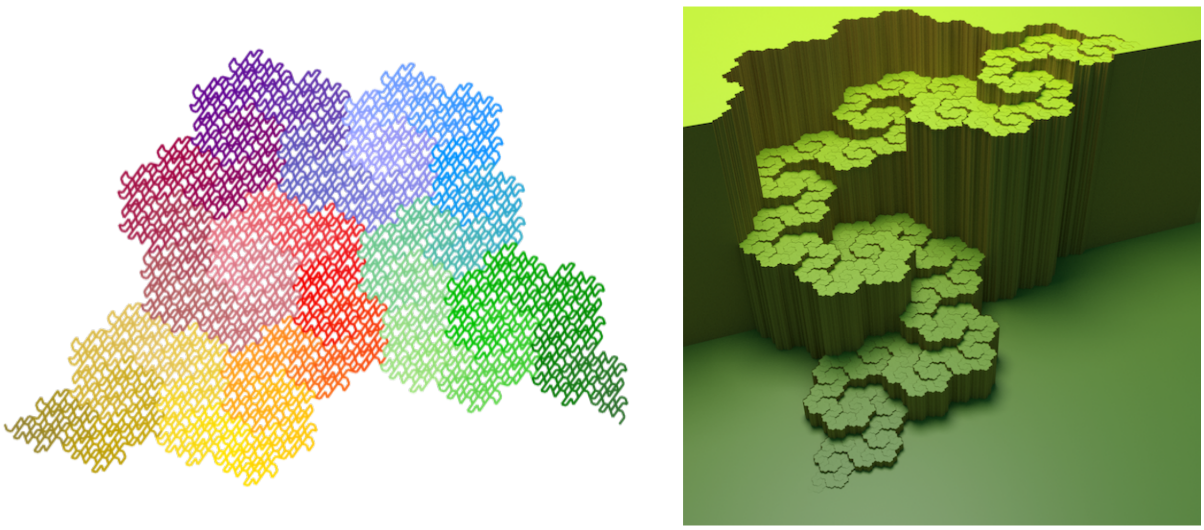

Figure 2: Left: a more detailed sketch of the Rauzy triangle curve. Right: the Rauzy triangle curve visualised as an ascending path [4].