Three pretty simple plane-filling curves

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Sometimes, when trying to find plane-filling curves that optimize certain useful qualities, I accidentally stumble on beautiful curves that I had not seen before. This document shows three such curves.

One of the many ways in which plane-filling curves can be described is the following, used by Ventrella [2]. One starts with a line segment; the segment is directed (it has a head and a tail) and oriented (its left and right side are distinct: one side is marked). We indicate the direction and orientation by a arrow head on the marked side of the head. A replacement rule describes how such a line segment is replaced by a polyline of directed, oriented line segments. Arrow heads on the segments of the polyline help to determine how each segment is obtained from the original line segment by scaling, translation, rotation, and/or reflection (see Figure 1 for an example). Applying the replacement rule recursively to each line segment results in a fractal curve. If the sum of the squared lengths of the segments of the replacing polyline is equal to the squared length of the original line segment, then the fractal curve has dimension two and, if it does not overlap itself too much, it will be a plane-filling curve.

Ventrella describes several examples that can be obtained by replacing a line segment by a sequence of only three line segments, each a factor $\sqrt{3}$ shorter than the original line segment, and such that all line segments on the same level of recursion lie on the edges of a regular grid of equilateral triangles. One example produces Peano's original space-filling curve [1], stretched by a factor $\sqrt{3}$ in one dimension. The replacement rule and one step of recursive application is shown in Figure 1a. However, when we expand the recursion further, we would find that on any level of recursion, the curve traverses most edges of a triangular grid and visits most vertices twice—so that we cannot tell from the drawing what course the curve takes. In the case of the Peano curve, this problem becomes apparent already on the second level of refinement, shown in Figure 1b. To alleviate this problem, we do not draw the line segments that define the curve, but we draw a "sketch" of each segment—which is really just another line segment or polyline that avoids the end points of the edge (see Figure 1c). Connecting the sketches of all segments results in a sketch of the plane-filling curve. A suitable choice of sketches results in the more familiar image of Peano's plane-filling curve as shown in Figure 1d; a colour gradient is clarify in what order regions of the plane are traversed. Note that the region of the plane that is filled by the curve is a simple rectangle.

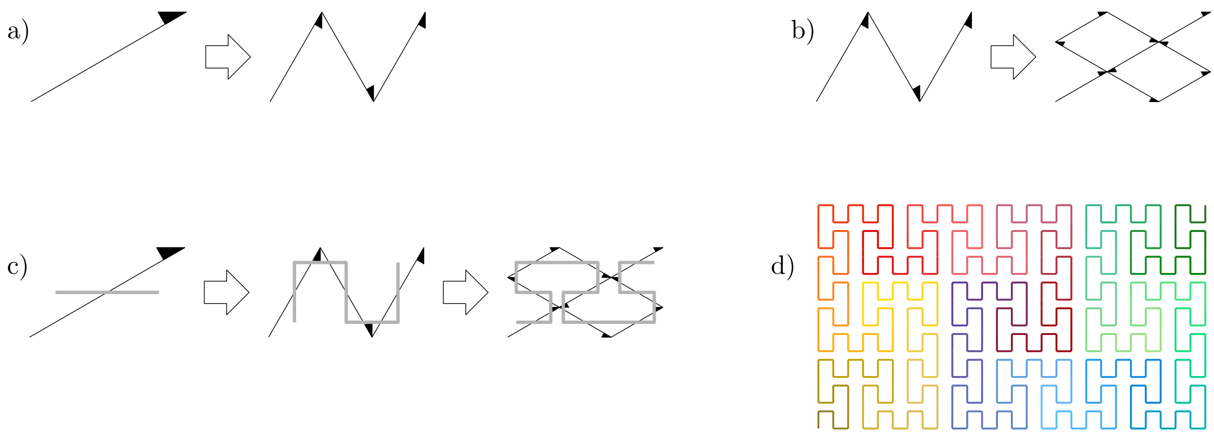

Figure 1: A definition and sketch of the Peano curve, stretched by a factor $\sqrt{3}$ in the horizontal dimension.

So far, nothing new. But what happens if you flip a bit in the computation for the Peano curve, and reflect each of the three segments of the polyline in the definition of Figure 1a? The surprising result is shown in Figure 2: we get a curve that fills a region that looks like a spiralling and twisting string of dancers that is composed of three interlocking smaller copies of itself. Note that on any level of recursion, this curve may visit edges of the grid twice (and vertices up to six times), but the two traversals of an edge differ in orientation (reflected or not reflected) and will, on deeper levels of recursion, expand to filling different regions in the plane. Therefore the edge sketch is chosen to be asymmetric with respect to the line containing the edge, so that the sketches of the reflected and the non-reflected traversals of the same edge do not coincide in the drawing.

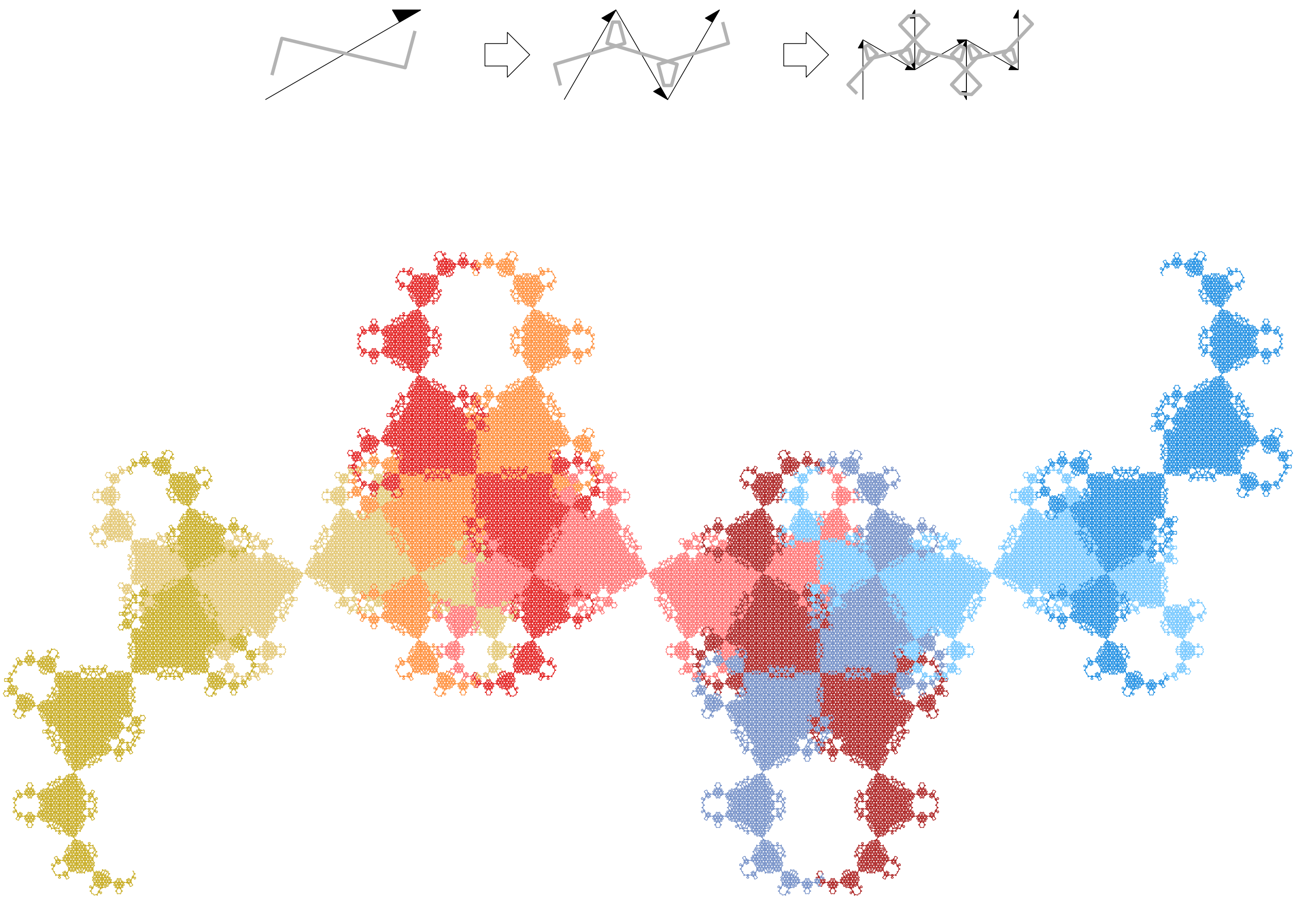

Figure 2: A definition and sketch of a plane-filling curve that fills an area that looks like a string of dancers that is composed of three smaller copies of itself.

Two more pretty curves with almost equally simple definitions (consisting of only three parts) are shown in Figure 3.

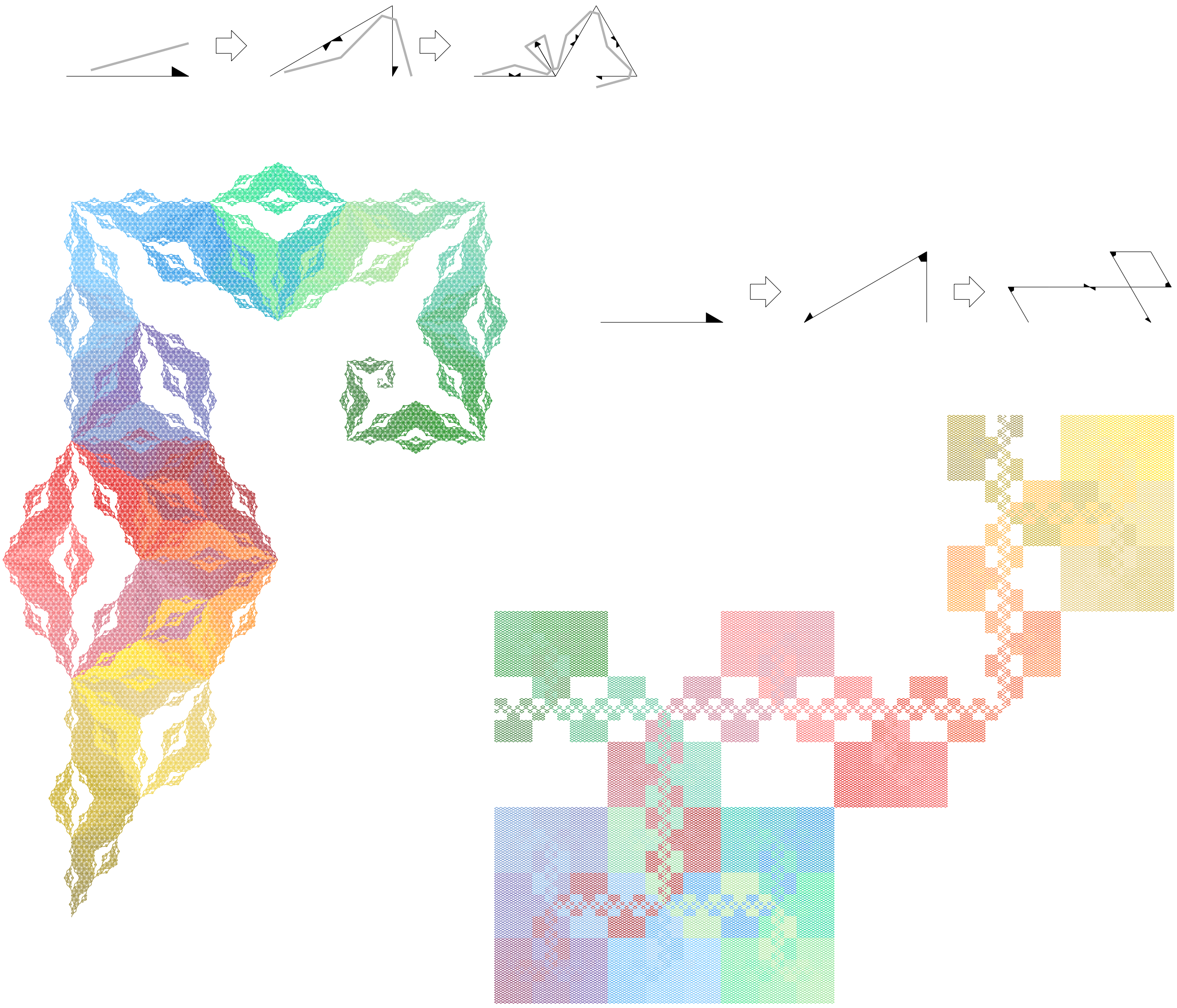

Figure 3: Left: a definition and sketch of a plane-filling curve that fills an area with a fuzzy (watery?) boundary that is composed of three smaller copies of itself. The colour sketch is rotated counterclockwise by 60 degrees. Right: A definition and sketch of a plane-filling curve that fills an area that is composed of three smaller copies of itself, which lock together along a zipper pattern. In this case, the sketch simply consists of edges of the grid. The colour sketch is rotated clockwise by 150 degrees.

References

[1] G. Peano. Sur une courbe, qui remplit toute une aire plane. *Math. Ann.*, 36(1):157–160 (1890)

[2] J. Ventrella. *Brain-filling curves—A fractal bestiary.* Eyebrain Books (2012).